Episode 14

Forced Vibrations Part 1
Excitation by External Forces

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Forced Vibrations

1. Damped harmonic oscillator subjected to external force
2. Examples
3. Using forced vibrations to measure natural frequency and damping factor
4. Forced vibration of systems with several DOF (optional – not covered in homeworks/exams)

5.6 Forced Vibrations

General Problem

Given : [k, m, c] or ζ, ω_n

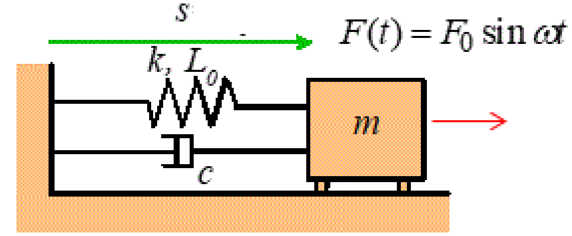
Harmonic excitation

F_0 or Y_0, frequency ω

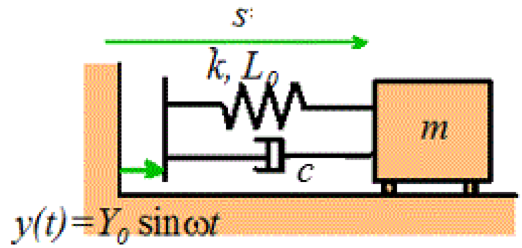
Initial conditions (sometimes)

Find $s(t)$

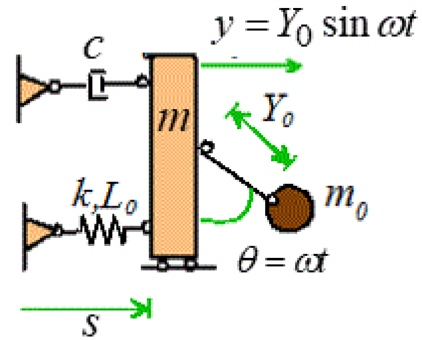
Externally applied force



Base Excitation



Unbalanced rotor/inertial excitation

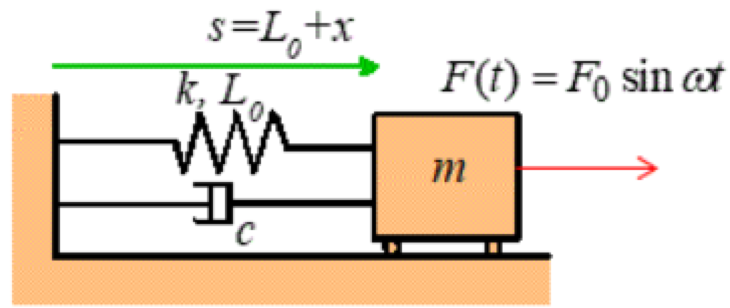


Consider each type of forcing in turn

5.6.1 Externally forced damped harmonic oscillator

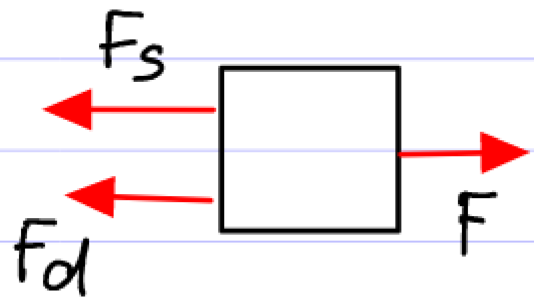
Canonical forced vibration problem: The spring mass system is released with velocity v_0 from position s_0 at time *t=0* . It is subjected to a force $F(t) = F_0 \sin \omega t$

Find *s(t)* .



Approach: (1) EOM ; (2) Tables

FBD



$$\underline{F} = m\underline{a} \Rightarrow F - F_s - F_d = m\frac{d^2s}{dt^2}$$

Spring / Damper $\quad F_s = k(s - L_0) \quad F_d = c\frac{ds}{dt}$

Hence

$$\boxed{\frac{m}{k}\frac{d^2s}{dt^2} + \frac{c}{k}\frac{ds}{dt} + \cancel{k}s = \cancel{k}L_0 + \frac{1}{k}F(t)}$$

List of standard ODEs for vibration problems

Case I $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+x=C$

Case II $\dfrac{1}{\alpha^2}\dfrac{d^2x}{dt^2}-x=-C$

Case III $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt}+x=C$

Case IV $\boxed{\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt}+x=C+KF(t) \text{ with } F(t)=F_0\sin\omega t}$

Case V $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt}+x=C+K\left(y+\dfrac{2\varsigma}{\omega_n}\dfrac{dy}{dt}\right)$ with $y(t)=Y_0\sin\omega t$

Case VI $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt}+x=C-\dfrac{K}{\omega_n^2}\dfrac{d^2y}{dt^2}$ with $y(t)=Y_0\sin\omega t$

Case VII $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\zeta}{\omega_n}\dfrac{dx}{dt}+x=K\left(\dfrac{\lambda^2}{\omega_n^2}\dfrac{d^2y}{dt^2}+\dfrac{2\zeta}{\omega_n}\dfrac{dy}{dt}+y\right)$ with $y(t)=Y_0\sin\omega t$

Our eq: $\dfrac{m}{k}\dfrac{d^2s}{dt^2}+\dfrac{c}{k}\dfrac{ds}{dt}+S=L_0+\dfrac{1}{k}F(t)$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C+K F(t)$$

Hence $\boxed{\omega_n=\sqrt{\dfrac{k}{m}} \qquad \zeta=\dfrac{c}{2\sqrt{km}} \qquad K=\dfrac{1}{k}}$

Solution to Case IV (From pdf on website)

Equation $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF_0\sin(\omega t)$ Initial Conditions $x=x_0$ $\frac{dx}{dt}=v_0$ $t=0$

Full Solution $x(t)=C+\boxed{x_h(t)}+\boxed{x_p(t)}$ Solution has 2 parts

Steady state part (particular integral) $x_p(t)=X_0\sin(\omega t+\phi)$

Important!

$$X_0=\frac{KF_0}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

Transient part (complementary integral)

Overdamped $\varsigma>1$ $x_h(t)=\exp(-\varsigma\omega_n t)\left\{\frac{v_0^h+(\varsigma\omega_n+\omega_d)x_0^h}{2\omega_d}\exp(\omega_d t)-\frac{v_0^h+(\varsigma\omega_n-\omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$

Critically Damped $\varsigma=1$ $x_h(t)=\left\{x_0^h+\left[v_0^h+\omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$

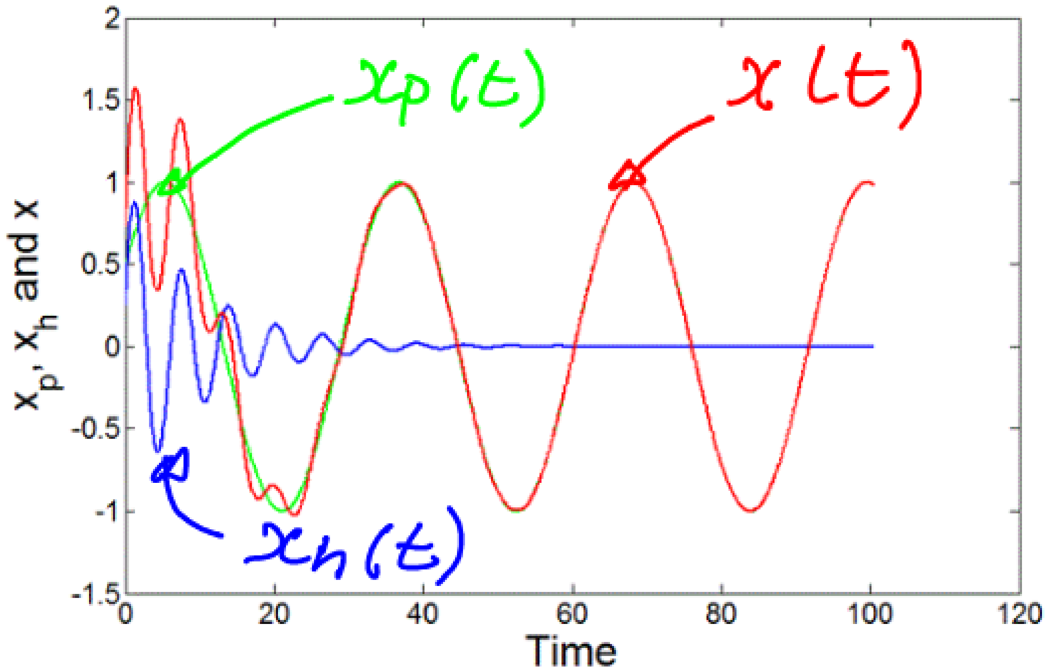
Underdamped $\varsigma<1$ $x_h(t)=\exp(-\varsigma\omega_n t)\left\{x_0^h\cos\omega_d t+\frac{v_0^h+\varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$

$\omega_d=\omega_n\sqrt{|1-\varsigma^2|}$ $x_0^h=x_0-C-x_p(0)=x_0-C-X_0\sin\phi$ $v_0^h=v_0-\left.\frac{dx_p}{dt}\right|_{t=0}=v_0-X_0\omega\cos\phi$

Less important...

Understanding forced vibration solution

Solution has 2 parts $x = x_h + x_p$



Steady State Solution

$x_p(t) = X_0 \sin(\omega t + \phi)$

X_0 : Amplitude
Depends on ω, ζ, ω_n
F_0, K

ω: same as force frequency

Transient Solution

$x_h(t)$: Free Vibration response
Decays with time

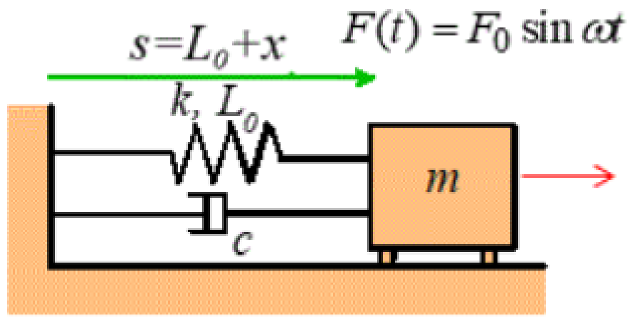
$x_h(t) \to 0$
as $t \to \infty$

Steady state solution for externally forced system

Steady state solution to $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF_0\sin(\omega t) \qquad \omega=2\pi/T$$

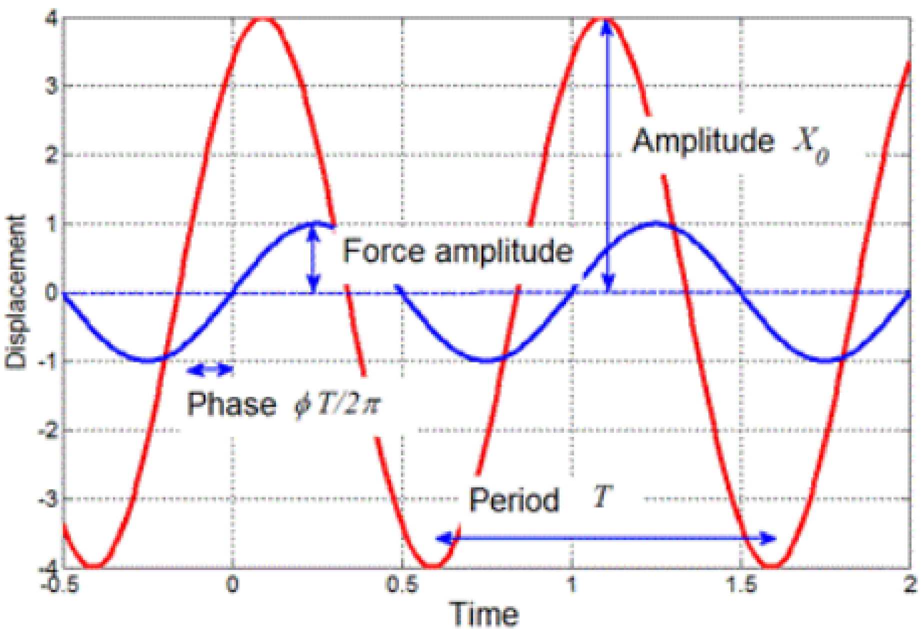
$$\omega_n=\sqrt{\frac{k}{m}} \qquad \zeta=\frac{c}{2\sqrt{km}} \qquad K=\frac{1}{k}$$

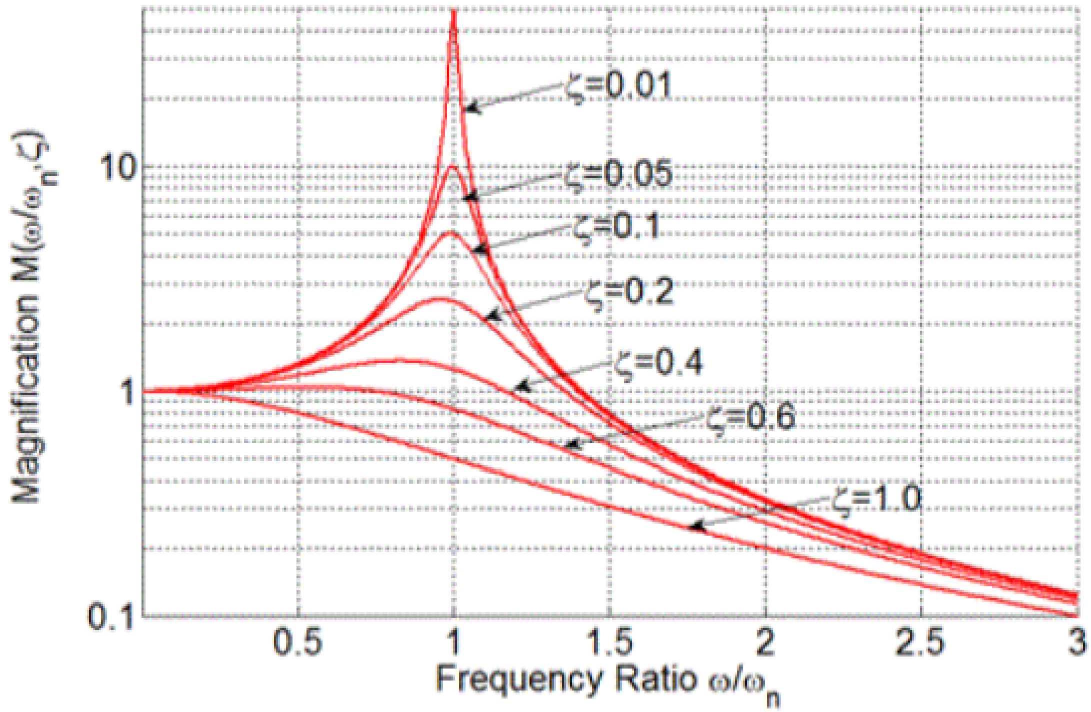
$$\boxed{x(t)=X_0\sin(\omega t+\phi)}$$



$$\boxed{X_0=KF_0M(\omega/\omega_n,\zeta) \qquad M=\frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}}$$

$$\phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$





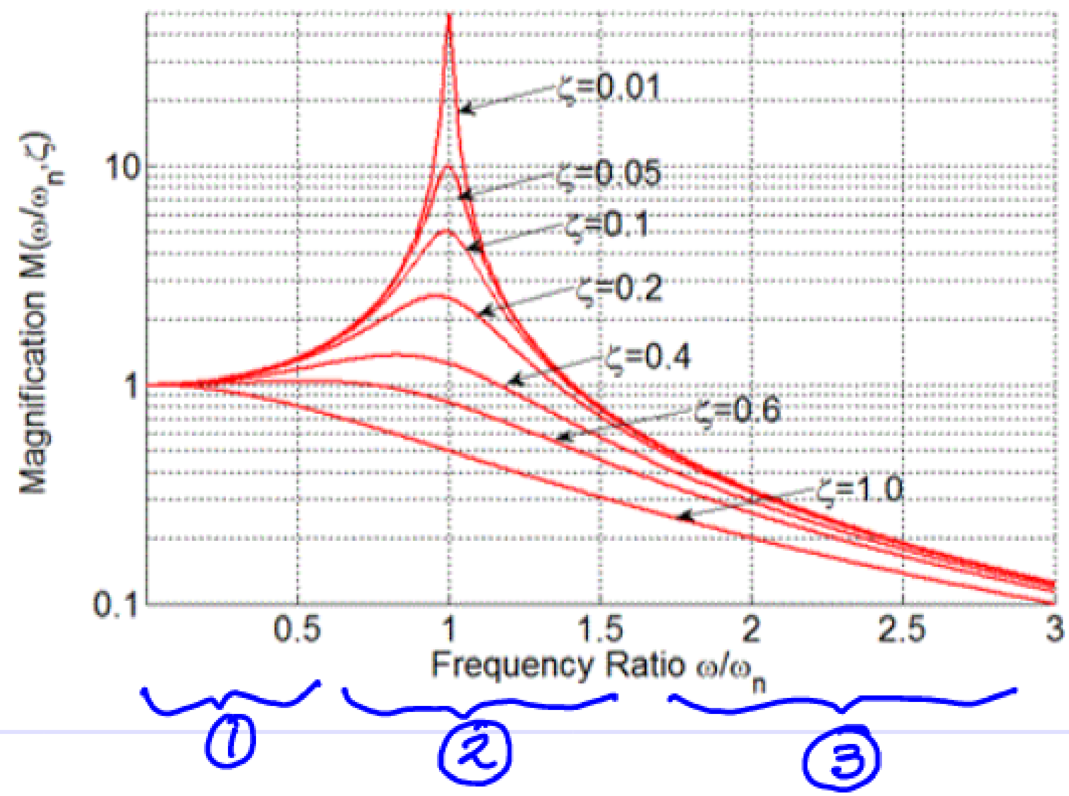
Understanding steady-state vibrations

$$x_p(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = K F_0 \, M\left(\frac{\omega}{\omega_n}, \zeta\right)$$

$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

$$K = 1/k$$



① $\omega \ll \omega_n \Rightarrow M \approx 1 \Rightarrow X_0 = F_0/k$ (static)

② $\omega \approx \omega_n \Rightarrow M \approx \frac{1}{2\zeta} \Rightarrow X_0 = \frac{F_0}{2k\zeta}$ "Resonance" Big vibrations

③ $\omega \gg \omega_n \Rightarrow M \sim \frac{\omega_n^2}{\omega^2}$ $X_0 = \frac{F_0}{m\omega^2}$ Vibrations decrease

Solving the case IV vibration equation

Solve: $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C+KF_0\sin\omega t \qquad x=x_0 \qquad \frac{dx}{dt}=v_0 \qquad t=0$$

Rewrite as: $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C-KF_0\frac{i}{2}\left(e^{i\omega t}-e^{-i\omega t}\right)$$

Guess solution: $$x(t)=C+x_p(t)+x_h(t) \qquad x_p(t)=-KF_0\frac{i}{2}\left(B_1e^{i\omega t}-B_2e^{-i\omega t}\right) \qquad x_h=Ae^{\lambda t}$$

Substitute into ODE:

$$-KF_0\frac{i}{2}\left(\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2\zeta\omega}{\omega_n}i\right)B_1e^{i\omega t}-\left(1-\frac{\omega^2}{\omega_n^2}-\frac{2\zeta\omega}{\omega_n}i\right)B_2e^{-i\omega t}\right)+\left(\frac{\lambda^2}{\omega_n^2}+\frac{2\zeta\lambda}{\omega_n}+1\right)Ae^{\lambda t}=-KF_0\frac{i}{2}\left(e^{i\omega t}-e^{-i\omega t}\right)$$

Hence $$B_1\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2\zeta\omega}{\omega_n}i\right)=1 \qquad B_2\left(1-\frac{\omega^2}{\omega_n^2}-\frac{2\zeta\omega}{\omega_n}i\right)=1 \qquad \left(\frac{\lambda^2}{\omega_n^2}+\frac{2\zeta\lambda}{\omega_n}+1\right)Ae^{\lambda t}=0$$

$$B_1=\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2\zeta\omega}{\omega_n}i\right)^{-1}=M(\omega/\omega_n,\zeta)e^{i\phi} \qquad B_2=\left(1-\frac{\omega^2}{\omega_n^2}-\frac{2\zeta\omega}{\omega_n}i\right)^{-1}=M(\omega/\omega_n,\zeta)e^{-i\phi}$$

$$M(\omega/\omega_n,\zeta)=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \qquad \phi=\tan^{-1}\frac{-2\zeta\omega/\omega_n}{(1-\omega^2/\omega_n^2)}$$

Steady state solution $$x_p(t)=-\frac{i}{2}KF_0M(\omega/\omega_n,\zeta)\left(e^{i(\omega t+\phi)}-e^{-i(\omega t+\phi)}\right)$$
$$=KF_0M(\omega/\omega_n,\zeta)\sin(\omega t+\phi)$$

Solving the case IV vibration equation

We have $x(t) = C + x_p(t) + x_h(t)$ with $x_p(t) = KF_0 M(\omega/\omega_n, \zeta)\sin(\omega t + \phi)$

Transient solution $x_h = Ae^{\lambda t}$ with $\left(\frac{\lambda^2}{\omega_n^2} + \frac{2\zeta\lambda}{\omega_n} + 1\right)Ae^{\lambda t} = 0$

Roots $\lambda = -\zeta\omega_n \mp \omega_n\sqrt{\zeta^2 - 1}$

Initial conditions $x_h(0) = x_0 - C - KF_0 M(\omega/\omega_n, \zeta)\sin\phi$

$$\left.\frac{dx_h}{dt}\right|_{t=0} = v_0 - KF_0 M(\omega/\omega_n, \zeta)\omega\cos\phi$$

Identical to damped free vibration problem – use previous solution

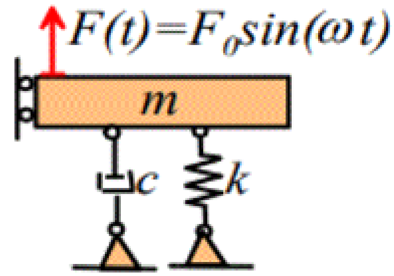
Overdamped $\zeta > 1$ $x_h(t) = \exp(-\zeta\omega_n t)\left\{\frac{v_0^h + (\zeta\omega_n + \omega_d)x_0^h}{2\omega_d}\exp(\omega_d t) - \frac{v_0^h + (\zeta\omega_n - \omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$

Critically Damped $\zeta = 1$ $x_h(t) = \left\{x_0^h + \left[v_0^h + \omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$

Underdamped $\zeta < 1$ $x_h(t) = \exp(-\zeta\omega_n t)\left\{x_0^h\cos\omega_d t + \frac{v_0^h + \zeta\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$

$\omega_d = \omega_n\sqrt{|1-\zeta^2|}$ $x_0^h = x_0 - C - x_p(0) = x_0 - C - KF_0 M(\omega/\omega_n, \zeta)\sin\phi$ $v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - KF_0 M(\omega/\omega_n, \zeta)\omega\cos\phi$

5.6.2 Example: A structure is idealized as a spring-mass system with the following properties



1. Mass 10000kg
2. Stiffness *k*: 100 kN/m
3. Dashpot coefficient 6 kNs/m

Wind loading subjects the structure to a harmonic force with amplitude 500N and frequency 5 Hz. Find the amplitude of vibration.

Approach: use formulas $X_0 = K F_0 M$

$K = 1/k = 1/100 \times 10^3$ $\omega_n = \sqrt{k/m} = 3.162$ rad/s

$\zeta = c/(2\sqrt{km}) = 0.1$

$F_0 = 500N$ $\omega = 10\pi$ rad/s

$$M = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} = 0.01$$

Hence $\boxed{X_0 = \frac{500}{100 \times 10^3} \times 0.01 = 51 \mu m}$

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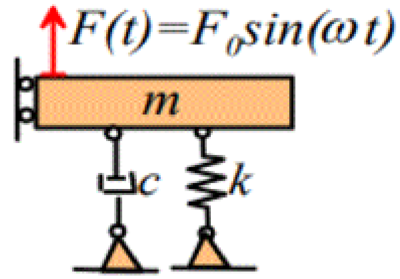

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$\bar{K} = 1/k = 1/100\times10^3$ $\quad \omega_n = \sqrt{k/m} = 3.162$ rad/s

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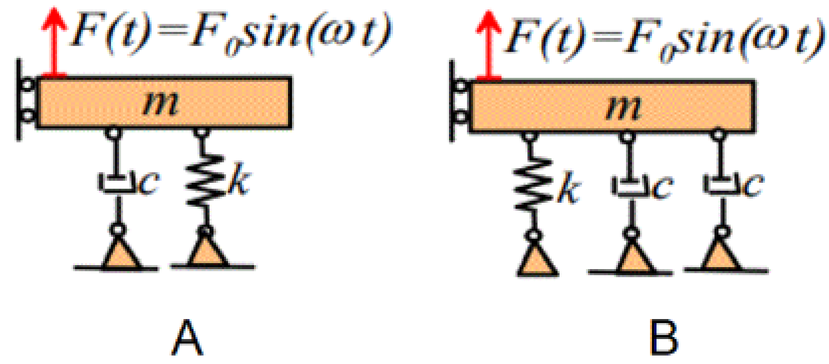
$$M = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\,\omega/\omega_n)^2}} = 0.01$$

Hence $\boxed{X_0 = \frac{500}{100\times10^3} \times 0.01 = 51\,\mu m}$

5.6.3 Example: Both systems in the figure are subjected to a force with amplitude 1 kN and frequency equal to the undamped natural frequency of the system.
The amplitude of vibration of system B is 1mm.
What is the vibration amplitude of system A?



Formula $\bar{X}_0 = \bar{K} F_0 M$ For $\omega=\omega_n$ $M = 1/2\zeta$

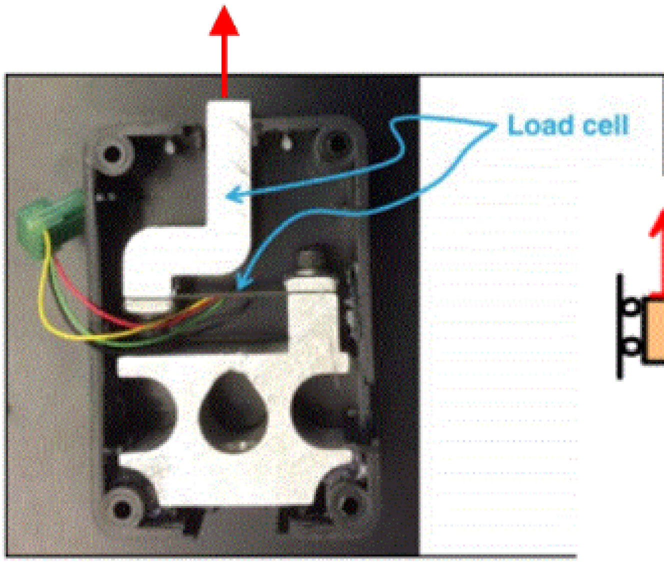
For both A & B $\bar{K} = 1/k$ $\omega_n = \sqrt{k/m}$

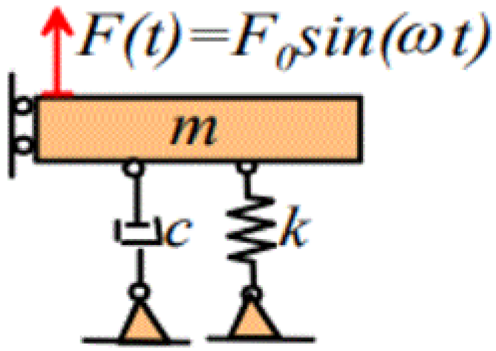
System B: $\zeta_B = 2c/2\sqrt{km}$

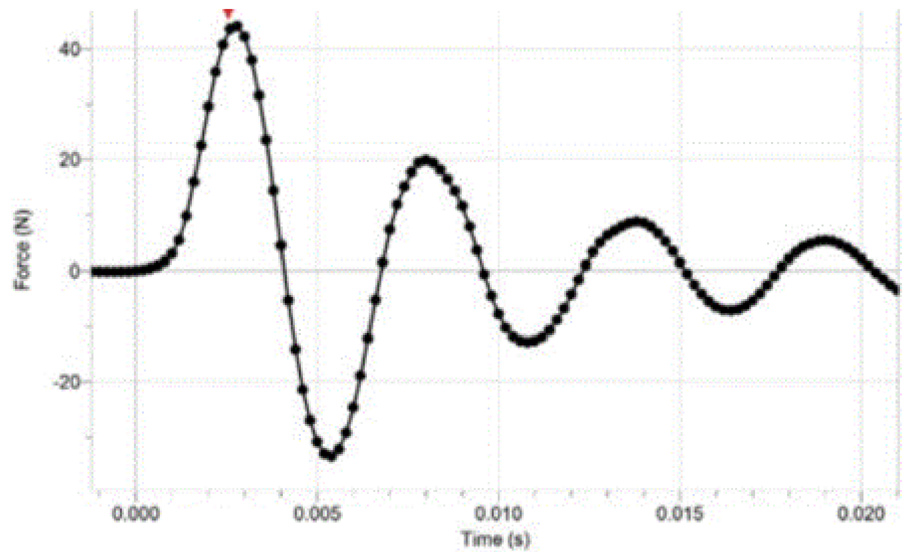
System A: $\zeta_A = c/2\sqrt{km}$

Hence $X_0^A / X_0^B = \zeta_B / \zeta_A = 2$

$\Rightarrow \boxed{\bar{X}_0^A = 2\,mm}$







5.6.4 Example: A force sensor behaves like a spring-mass system. The force is determined by measuring the extension x of the spring and converting the measurement to force as *F=kx*

A static measurement shows *k=84.9 kN/m*

The figure shows the results of an impulse test on the force sensor

(a) Find values for *m* and *c* for the sensor

(b) Suppose the sensor is subjected to a 10N harmonic force. Calculate the reading on the sensor for frequencies of (i) 10Hz and (ii) 150 Hz

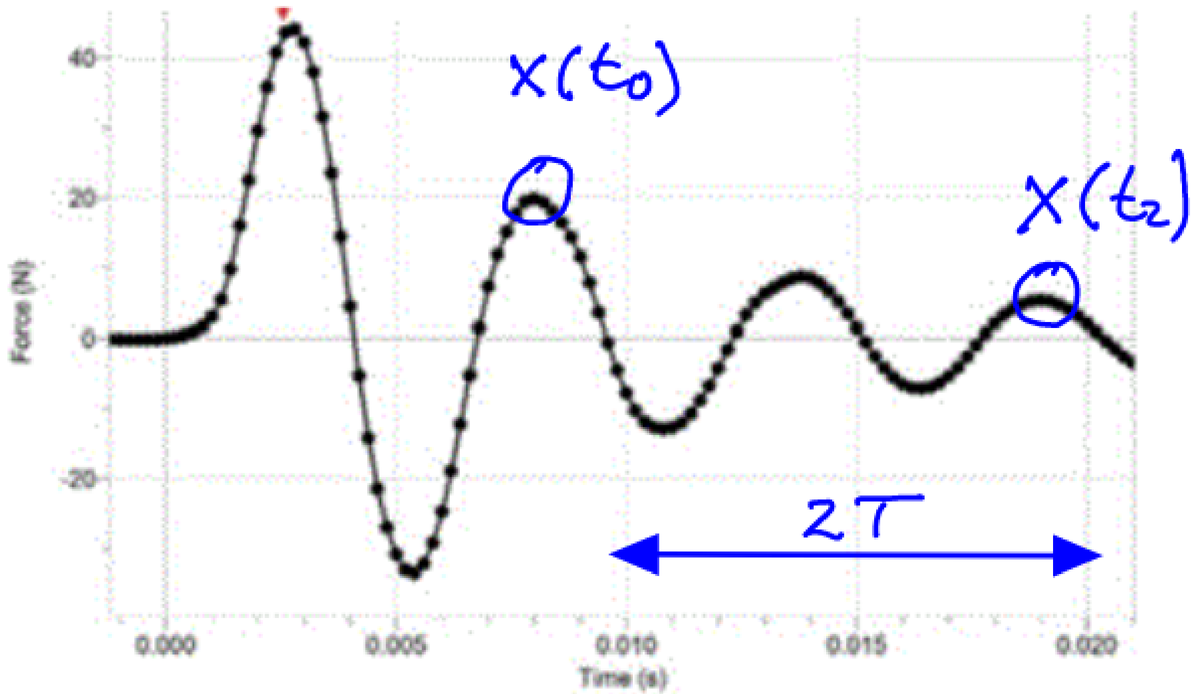
(c) Find the 'bandwidth' of the sensor (the frequency when the sensor overestimates the force by a factor of $\sqrt{2}$)

(a) Find $[\zeta, \omega_n, m, c]$ using method from L13

(b) Note force reading amplitude is $k\underline{X}_0$ Use formulas for $\underline{X}_0$

(c) Find value of ω when $k\underline{X}_0 / F_0 = \sqrt{2}$

(a) Find m, c



Period: $2T = 0.011$

$\Rightarrow T = 0.0055$

$$\delta = \frac{1}{n}\log\left(\frac{x(t_0)}{x(t_n)}\right)$$

$$= \frac{1}{2}\log\left(\frac{20}{5}\right) = 0.69$$

$$\omega_n = \frac{\sqrt{4\pi^2+\delta^2}}{T} = 1.15\times10^3 \text{ rad/s}$$

$$\zeta = \delta/\sqrt{4\pi^2+\delta^2} = 0.11$$

$$\omega_n = \sqrt{k/m} \Rightarrow \boxed{m = k/\omega_n^2 = 0.064 \text{ kg}}$$

$$\zeta = c/2\sqrt{km} \Rightarrow \boxed{c = 2\sqrt{km}\,\zeta = 16.2 \text{ Ns/m}}$$

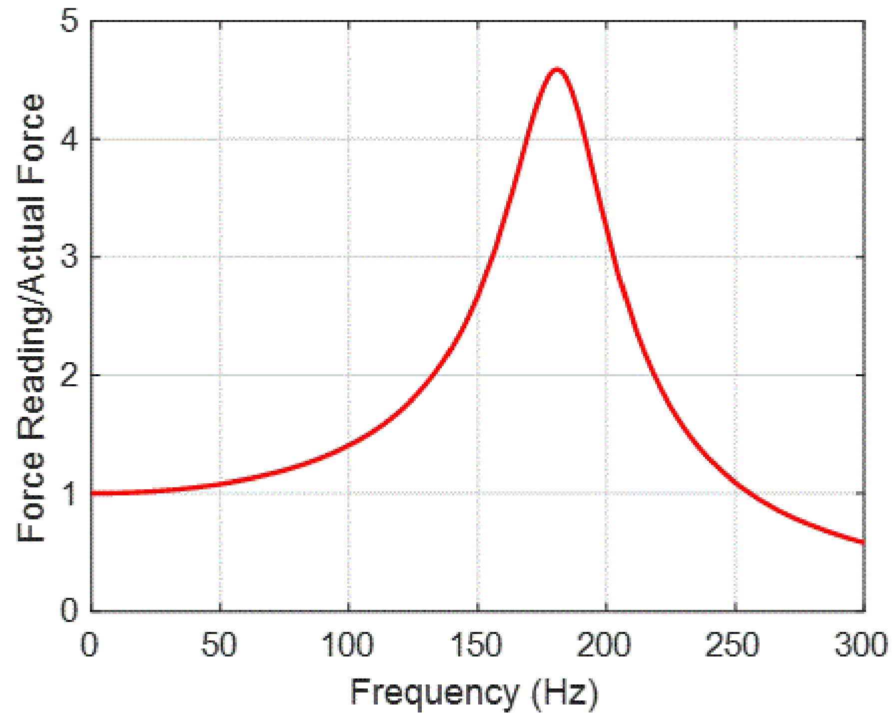
(b) Find force reading for $F_0 = 10\,N$ and $f = 10\,Hz$ and $f = 150\,Hz$

Force reading $F_R = k X_0$

Recall $X_0 = \frac{1}{k} F_0 \, M\left(\frac{\omega}{\omega_n}, \zeta\right)$

$\Rightarrow F_R = F_0 . M$

$$M = \frac{1}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$



Use ω_n, ζ from (a), note $\omega = 2\pi f$

$\boxed{\text{At } 10\,Hz \quad M = 1 \Rightarrow F_R = 10\,N}$

$\boxed{\text{At } 150\,Hz \quad M = 2.67 \Rightarrow F_R = 26.7\,N}$

(c) find "bandwidth" (ω when $F_R/F_0 = \sqrt{2}$)

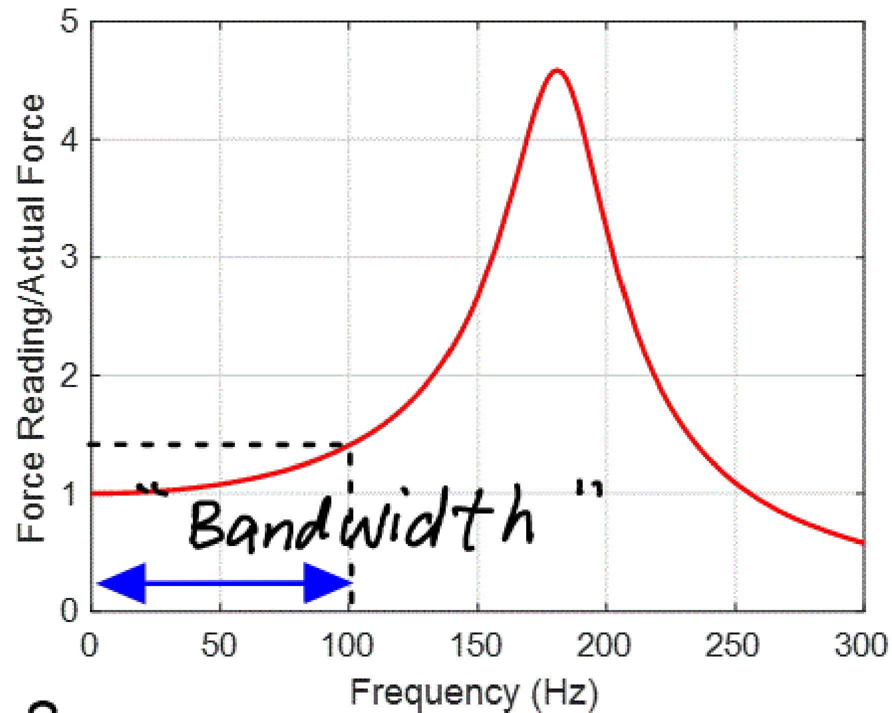
Recall $F_R = F_0 M \Rightarrow$ at ω_{BW} $M = \sqrt{2}$

$$M = \sqrt{2} \Rightarrow \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} = \sqrt{2}$$

Let $r = (\omega/\omega_n)^2$

$$\Rightarrow (1-r)^2 + (2\zeta)^2 r = 1/2$$

$$\Rightarrow r^2 + ((2\zeta)^2 - 2)r + 1/2 = 0$$



For $\zeta = 0.11$ roots are $r = 0.303$
$r = 1.65$

Hence $\boxed{\omega_{BW} = \sqrt{0.303}\,\omega_n = 633 \text{ rad/s}}$
or $\boxed{f_{BW} = \omega_{BW}/2\pi = 100.7 \text{ Hz}}$

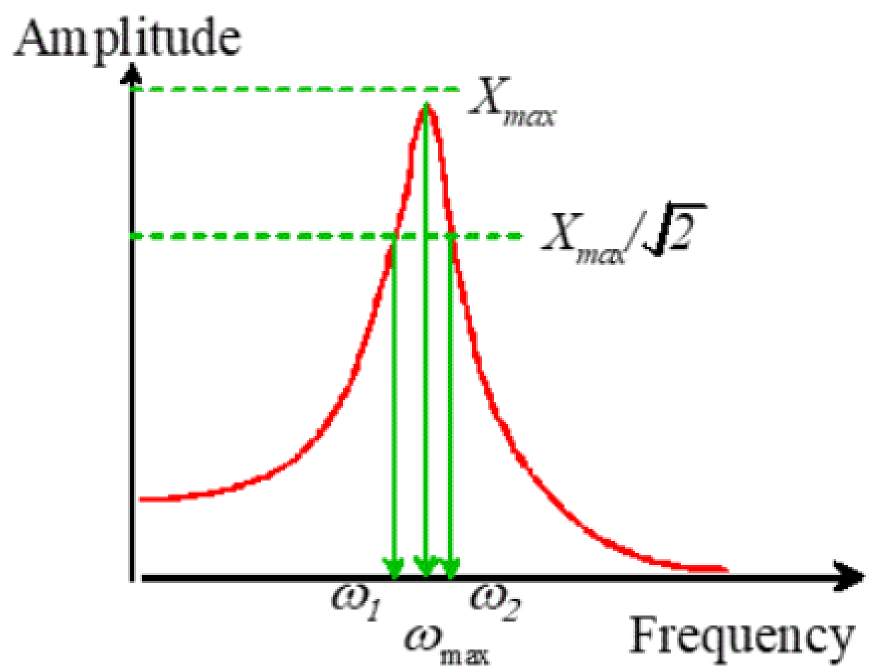
5.6.5 Using forced vibrations to measure ω_n & ζ

Procedure: measure vibration amplitude for a range of ω near ω_n for a vibration mode (eg swept sin test)

Find ω_1, ω_2 as shown

Define "Q factor" $Q = \dfrac{\omega_{max}}{\omega_2 - \omega_1}$

"Bandwidth" $\Delta\omega = \omega_2 - \omega_1$



Formulas:

$$\boxed{\zeta \approx \frac{1}{2Q}}$$

$$\boxed{\omega_n \approx \omega_{max}}$$

<u>Proof:</u> For $\zeta \ll 1$ $\quad X_{0max} = \dfrac{K F_0}{2\zeta}$ at $\omega = \omega_n$

Hence at $\omega = \omega_1$, $\omega = \omega_2$

$$\frac{K F_0}{2\zeta\sqrt{2}} = K F_0 M = \frac{K F_0}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

Let $r = (\omega/\omega_n)^2 \Rightarrow 8\zeta^2 = (1-r)^2 + (2\zeta)^2 r$

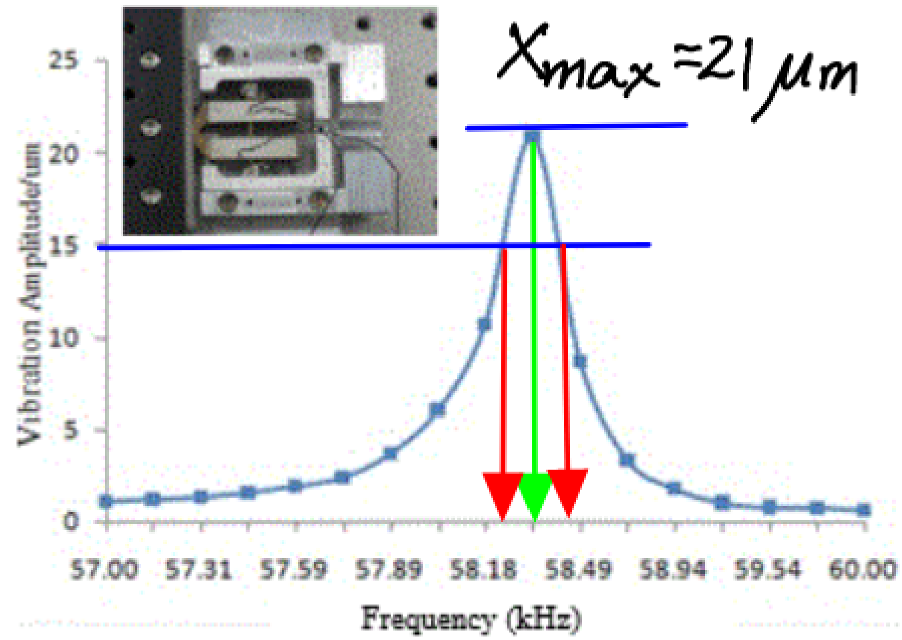
Roots $r = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1-\zeta^2}$

Hence $(\omega_2/\omega_n)^2 - (\omega_1/\omega_n)^2 = 4\zeta\sqrt{1-\zeta^2} \approx 4\zeta$

$$\Rightarrow (\omega_2 - \omega_1)\overbrace{(\omega_2 + \omega_1)}^{\approx 2\omega_n} = 4\omega_n^2\zeta$$

$$\Rightarrow \frac{\omega_2 - \omega_1}{2\omega_n} = \zeta \qquad \Rightarrow \frac{1}{2Q} = \zeta$$

Example: The figure shows the vibration amplitude of an ultrasonic motor as a function of frequency. Calculate ζ, ω_n for the vibration mode



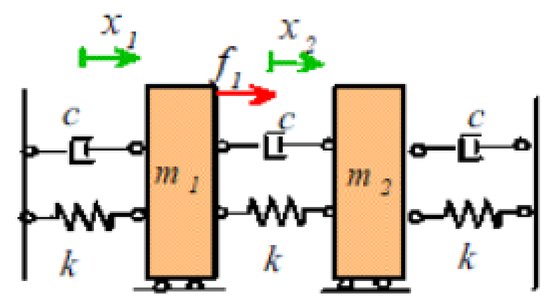
$f_{max} \sim 58.3\ kHz$

$X_{max} = 21\,\mu m \Rightarrow X_{max}/\sqrt{2} = 15\,\mu m$

$f_1 = 58.2\ kHz \qquad f_2 = 58.4\ kHz$

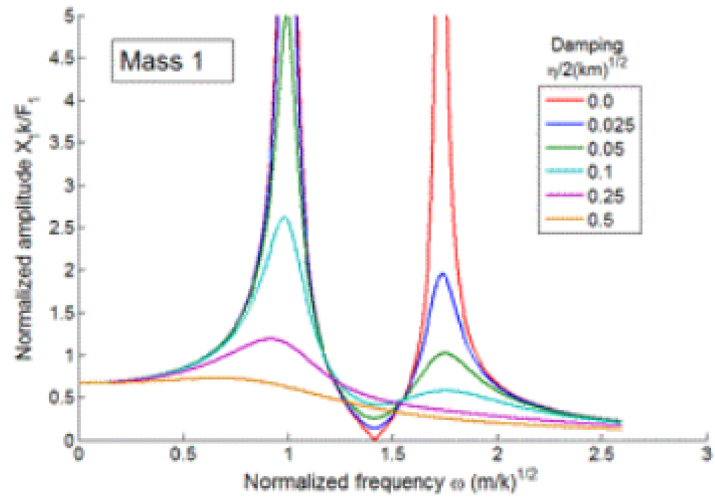
$$\Rightarrow \boxed{\begin{aligned} \zeta &= \frac{1}{2}\frac{(58.4 - 58.2)}{58.3} = 0.002 \\ \omega_n &= 2\pi f_{max} = 366 \times 10^3\ rad/s \end{aligned}}$$

5.6.6 Forced vibration of systems with several DOF

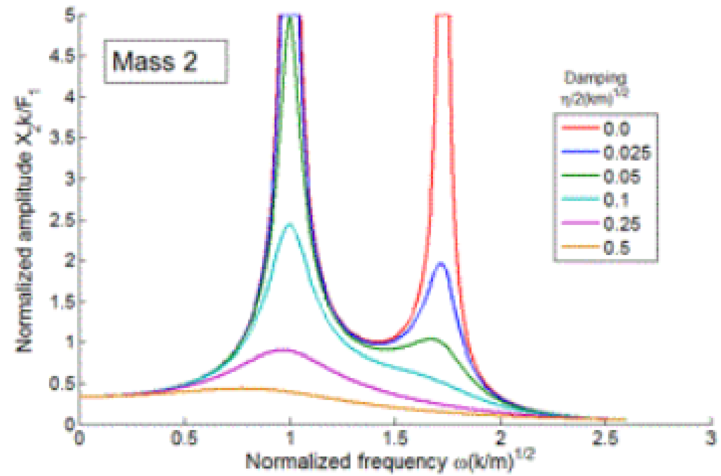


Observations from experiment & simulations

(1) Resonance occurs whenever forcing frequency is near a natural frequency



(2) There may be an "anti-resonance" between two resonant peaks

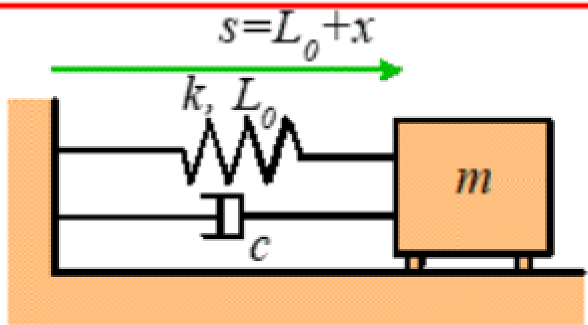


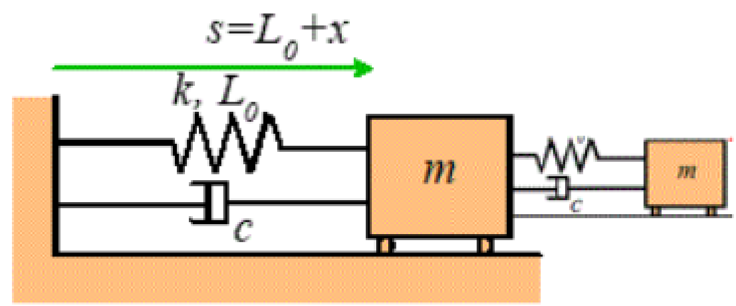
Engineering implications

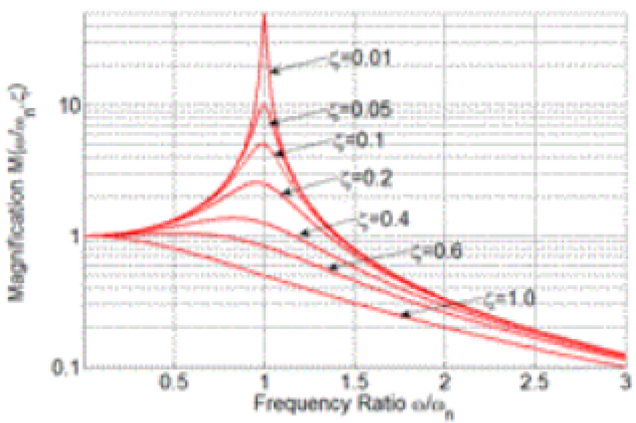
(1) To avoid resonance, ensure forcing frequency << lowest natural frequency

(2) Application of "anti-resonance": the "Tuned Mass Damper"

System with vibration problem → Add a 2nd mass & tune for anti-resonance







anti-resonance